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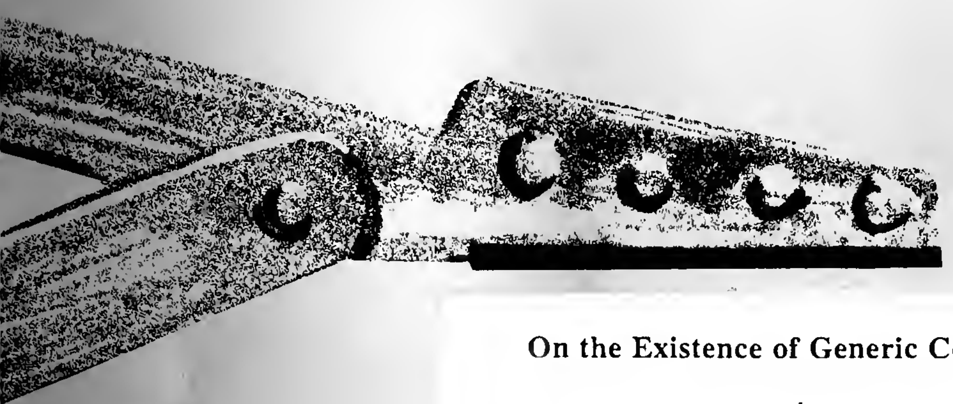

On the Existence of Generic Coordinates

by

Thomas Dubé

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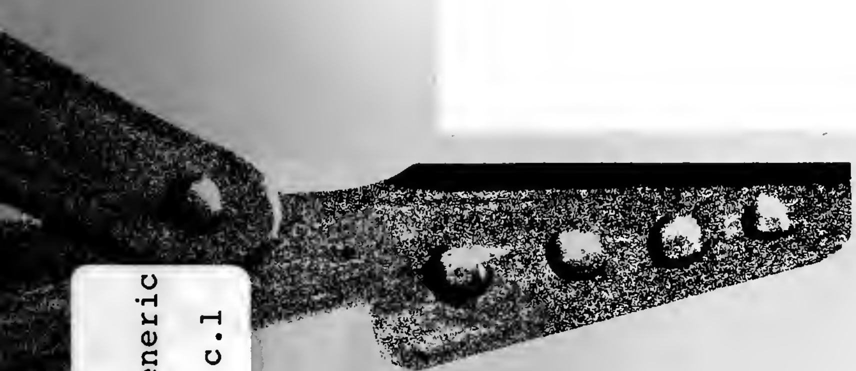

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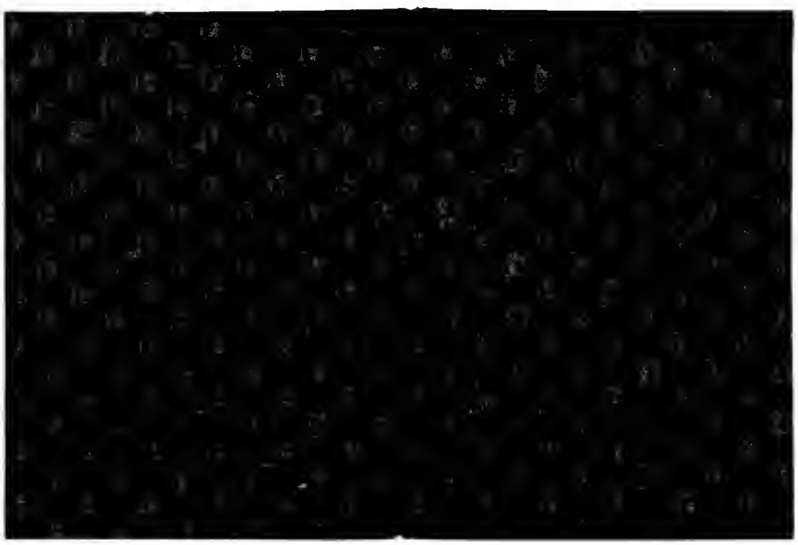

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On the Existence of Generic Coordinates

Thomas W. Dubé

June 12, 1989

Abstract

A well-known technique in computational mathematics is to change to a coordinate system which exhibits special attractive properties. A somewhat unusual case is the change to *generic coordinates* in proving a degree bound for Gröbner bases. Here, the generic coordinates exhibit the worst case behavior, and thus allow the construction of upper bounds.

A sketch of the existence of such coordinates was provided in Bayer's thesis. This report expands upon that sketch to provide a fully detailed proof that such a coordinate system always exists.

1 Introduction

Let $\mathcal{A} = K[x_1, \ldots, x_n]$ be a multi-variate polynomial ring with coefficients in a field K.
Definition: A monomial ideal I is said to be Borel-fixed if it is invariant under an upper-triangular linear change of coordinates. In practice, it is often convenient to use one of the equivalent definitions:

1. A monomial ideal I is Borel-fixed if for each pair of variables x_i, x_j such that $i < j$,
$$(I : x_j) \subseteq (I : x_i) .$$

2. A monomial ideal I is Borel-fixed if for every power product P and index $i < n$,
$$x_{i+1}P \in I \quad \text{implies} \quad x_iP \in I .$$

For a monomial $A = kx_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$, the degree of A will refer to the total-degree

$$\deg(A) \; = \; \sum_{i=1}^{n} a_i \; .$$

The above definitions of a Borel-fixed ideal can be sharpened to provide the following effective criteria for determining if an ideal is Borel-fixed.

Lemma 1 *Let I be an ideal generated by a set of monomials $F = \{f_1, \ldots, f_r\}$, and let d be the maximum of the degrees of the f_i. Then the following conditions are equivalent:*

1. *I is Borel-fixed.*
2. *For each power product P with $\deg(P) \leq d$*

$$x_{i+1}P \in I \quad \textit{implies} \quad x_i P \in I \; .$$

3. *For each generator f_k and index $j < n$:*

$$f_k \textit{ a multiple of } x_{j+1} \implies x_j x_{j+1}^{-1} f_k \in I \; .$$

Proof. The implication (1) $\implies$ (2) $\implies$ (3) is trivial. It remains to be proven that (3) $\implies$ (1).

Assume that condition (3) holds. For any $j < n$ and monomial P such that $x_{j+1}P \in I$, it must be shown that $x_j P \in I$. Since F is a monomial basis for I, $x_{j+1}P = f_k g$ for some f_k in the basis and monomial g. If g is a multiple of x_{j+1}, then $P \in I$ and hence $x_j P \in I$. Otherwise, f_k is a multiple of x_{j+1} and so be condition (3),

$$\begin{aligned} x_j x_{j+1}^{-1} f_k &\in I \\ x_j P = (x_j x_{j+1}^{-1} f_k) g &\in I \; . \end{aligned}$$

□

Definition: The reverse lexicographic ordering ($\underset{R}{>}$) is a total-ordering defined on the power products $\mathrm{PP}[x_1, \ldots, x_n]$ as follows: Let $A = x_1^{a_1}\cdots x_n^{a_n}$ and $B = x_1^{b_a}\cdots x_n^{b_n}$, then $A \underset{R}{>} B$ if

1. $\deg(A) \; > \; \deg(B)$, or

2. $deg(A) = \deg(B)$ and the last exponent at which A and B differ has $a_j < b_j$.

For a polynomial h, the $\underset{R}{>}$-greatest power product of a monomial of h is called the head term of h and is denoted by $\mathrm{Hterm}_R(h)$. For an ideal I, $\mathrm{Head}_R(I)$ is used to denote the ideal generated by $\{\mathrm{Hterm}_R(h) : h \in I\}$ and is called the head ideal of I.

In [Ba 82], [Gi 84], [MöMo 84] a key step in the analysis of Gröbner basis [Bu 85] complexity is the transformation to generic coordinates. [1] In this usage, *generic coordinates* for an ideal I refers to a coordinate system where the head ideal of I w.r.t the reverse lexicographic ordering is Borel-fixed. The term *generic* is used because after nearly any random change of coordinates this condition will occur.

[Ba 82] provides an informal proof that if the coefficient field K is infinite, then almost all linear upper triangular coordinate changes result in a transformation to generic coordinates. This same conclusion follows from [Ga 73], but there analytical methods are employed. This report expands on the sketch provided by Bayer, and provides a formal construction which shows that if the coefficient field is infinite, than a generic coordinate system can always be found.

2 Background: Extending the Field

This section is used to collect a few well-known properties of extended formal power series, and the consequences of extending the coefficient field of a polynomial ring. This information can be found in more detail in (among other sources) [ZaSa 60].

Definition: For a field K, the extended formal power series $K\langle y\rangle$ is a field whose elements are of the form $a = \sum_{j=-\infty}^{\infty} a_j y^j$. Where $a_j \in K$ and for each element a, there exists an $m \in \mathbf{Z}$ such that $a_j = 0$ for $j < m$. In other words, each $a \in K\langle y\rangle$ can be written as $a = \sum_{j=m}^{\infty} a_j y^j$ for some $m \in \mathbf{Z}$.

For $\mathcal{A} = K[x_1, \ldots, x_n]$ let $\mathcal{A}'$ denote the polynomial ring $(K\langle y\rangle)[x_1, \ldots, x_n]$. In $\mathcal{A}'$, each monomial consists of a power product in $\mathrm{PP}[x_1, \ldots, x_n]$ multiplied by a coefficient in $K\langle y\rangle$. That is, monomials have coefficients which

[1] [Du 89] provides a method of producing degree bounds which does not require a change of coordinates.

are power series in y.

For I a homogeneous ideal of $\mathcal{A}$, let I' denote the ideal generated by I in $\mathcal{A}'$. Then,

1. Let $F = \{f_1, \ldots, f_r\} \subset \mathcal{A}$ be any basis for I. Then F generates I' in $\mathcal{A}'$.

2. For $h \in \mathcal{A}'$ consider a decomposition of h of the form $h = \sum_{j=-\infty}^{\infty} y^j h_j$ for $h_j \in \mathcal{A}$. Then, $h \in I'$ if and only if for all j $h_j \in I$.

3. Let I_d denote the set $\{h \in I \mid \deg(h) = d\}$ and similarly let I'_d denote the set $\{h \in I' \mid \deg(h) = d\}$. Then $h \in I'_d$ if and only if h can be written as $h = \sum_{-\infty}^{\infty} y^j h_j$ with $h_j \in I_d$.

4. Let $\underset{A}{>}$ be any total ordering on the power products $\mathrm{PP}[x_1, \ldots, x_n]$, for example the reverse lexicographic ordering. Then,

$$\{\mathrm{Hterm}_A(h) \ : \ h \in I'\} \ = \ \{\mathrm{Hterm}_A(h) \ : \ h \in I\}$$

5. I and I' have the same Hilbert functions.

3 The Change of Coordinates

Definition: A homomorphism $\phi : \mathcal{A} \longrightarrow \mathcal{A}$ is called a linear change of coordinates if for every homogeneous polynomial h, $\phi(h)$ is also homogeneous and $\deg(\phi(h)) = \deg(h)$.

For any homomorphism ϕ:

1. If $F = \{f_1, \ldots, f_r\}$ is a basis for I, then $\{\phi(f_1), \ldots, \phi(f_r)\}$ is a basis for the ideal $J = \{\phi(h) \ : \ h \in I\}$.

2. If $h_1, \ldots, h_s$ are linearly independent, then so are $\phi(h_1), \ldots, \phi(h_s)$.

3. If ϕ is degree preserving, then for every homogeneous ideal I, the ideals I and $\phi(I)$ share the same Hilbert function.

Let ϕ_s denote the change of coordinates in the ring $\mathcal{A}'$ given by

$$\phi_s : \begin{cases} x_{s+1} \longrightarrow y x_s + x_{s+1} & \\ x_i \longrightarrow x_i & i \neq s+1 \ . \end{cases}$$

The inverse of this homomorphism is simply:

$$\phi_s^{-1} : \begin{cases} x_{s+1} & \longrightarrow & -yx_s + x_{s+1} \\ x_i & \longrightarrow & x_i & \qquad i \neq s+1 \ . \end{cases}$$

The analysis of this coordinate change will concentrate on the two variables x_s and x_{s+1}. To facilitate separating these two variables from the remaining ones, the following notation is introduced:

$$\overline{\mathrm{PP}}_s = \mathrm{PP}[x_1, \ldots, x_{s-1}, x_{s+2}, \ldots, x_n] \ .$$

In other words, $\overline{\mathrm{PP}}_s$ is the set of power products which do not involve the variables x_s and x_{s+1}. Let M be a monomial $x_s^u x_{s+1}^v P$, for $P \in \overline{\mathrm{PP}}_s$. Then,

$$\phi_s(M) = \sum_{j=0}^{v} \binom{v}{j} y^j x_s^{u+j} x_{s+1}^{v-j} P.$$

Note that the power products

$$x_s^{u+v} P,\ x_s^{u+v-1} x_{s+1} P,\ \ldots,\ x_{s+1}^{u+v} P$$

appear consecutively in reverse lexicographic order.

Lemma 2 *Let $h \in A'$, and let* $\mathrm{Hterm}(h) = x_s^u x_{s+1}^v P$ *where $P \in \overline{\mathrm{PP}}_s$. Then* $\mathrm{Hterm}(\phi_s(h))$ *is of the form $x_s^{u'} x_{s+1}^{v'} P$, where $u' + v' = u + v$.*

Proof. Since each monomial of h is $\underset{\mathrm{R}}{\leq} x_s^u x_{s+1}^v P$, the homomorphism ϕ_s can produce no monomial $\underset{\mathrm{R}}{>} x_s^{u+v} P$, so $x_s^{u+v} P \underset{\mathrm{R}}{\geq} \mathrm{Hterm}(\phi_s(h))$. Choose M' as the *least* monomial of h of the form $M' = x_s^{u-c} x_{s+1}^{v+c} P$, then M' must also be present in $\phi_s(h)$. One can therefore conclude

$$x_s^{u+v} P \underset{\mathrm{R}}{\geq} \mathrm{Hterm}(\phi_s(h)) \underset{\mathrm{R}}{\geq} x_{s+1}^{u+v} P \ .$$

□

Notation: For I an ideal of A' and $P \in \overline{\mathrm{PP}}_s$, let $N_P(I, d)$ denote the number of distinct head terms of the form $x_s^{d-c} x_{s+1}^c P$ which belong to polynomials in I. That is,

$$\begin{aligned} N_P(I,d) &= |\{c \ : \ x_s^{d-c} x_{s+1}^c P \in \mathrm{Head}(I)\}| \\ &= |\{c \ : \ x_s^{d-c} x_{s+1}^c \in \mathrm{Head}(I) : P\}| \ . \end{aligned}$$

Though it will not be used in this report, one might note that that is the Hilbert function $\varphi_J(d)$ for the ideal of $J = (\mathrm{Head}(I) : P) \cap K[x_s, x_{s+1}]$.

Lemma 3 *Let $I' \subseteq \mathcal{A}'$, and $J = \{\phi_s(h) \ : \ h \in I'\}$, then for any $P \in \overline{\mathrm{PP}}_s$ and d, $N_P(J, d) \ = \ N_P(I', d)$.*

Proof. Let $m = N_P(I', d)$. Choose $f_1, \ldots, f_m \in I'$ having distinct head terms of the form $x_s^{d-c} x_s^c P$. Since these m polynomials are linearly independent, the polynomials $\phi_s(f_1), \ldots, \phi_s(f_m)$ must also be linearly independent. Triangulating the $\phi_s(f_j)$, it is possible to create linear combinations $g_1, \ldots, g_m \in J$ of the form

$$\begin{aligned} g_i &= \sum_{j=1}^{m} a_{i,j} \phi_s(f_j) \\ &= \phi_s(\sum_{j=1}^{m} a_{i,j} f_j) \end{aligned}$$

such that $a_{i,j} \in K\langle y \rangle$, and the head terms of the g_i are distinct. Then $g_i = \phi_s(h_i)$ where $h_i = \sum_{j=1}^{m} a_{i,j} f_j$. Each h_i has a head term of the form $x_s^{d-c} x_{s+1}^c P$, and thus by the preceding lemma, $\mathrm{Hterm}(g_i)$ is therefore also of this form. And so each of the $g_1, \ldots, g_m \in J$ has a distinct head term of the form $x_s^{d-c} x_{s+1}^c P$ and therefore $N_P(J, d) \geq c$.

A symmetric argument employing the inverse homomorphism ϕ_s^{-1} shows that $N_P(I', d) \geq N_P(J, d)$, completing the proof of the lemma.
□

Lemma 4 *Let $I \in \mathcal{A}$, and I' be the ideal generated by I in $\mathcal{A}'$. Suppose that I' contains a polynomial h with $\mathrm{Hterm}(\phi_s(h)) \ = \ x_s^{d-c} x_{s+1}^c P$. Then $N_P(I, d) \geq c + 1$.*

Proof. Since $\mathrm{Hterm}(\phi_s(h)) = x_s^{d-c} x_{s+1}^c P$,

$$\phi_s(h) = \sum_{i=c}^{d} a_i x_s^{d-i} x_{s+1}^i P + \mathrm{L.O.T.}$$

Using the inverse transformation ϕ_s^{-1}, h can be written as:

$$\begin{aligned} h &= (\sum_{i=c}^{d} a_i x_s^{d-i} (x_{s+1} - y x_s)^i) P + \mathrm{L.O.T.} \\ &= \sum_{i=c}^{d} a_i (\sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} y^{i-j} x_s^{d-j} x_{s+1}^j) P + \mathrm{L.O.T.} \end{aligned}$$

The a_i are elements of $K\langle y\rangle$ and hence can be written as $a_i = \sum_{z=m}^{\infty} b_{i,z}y^z$. Collecting terms around powers of y

$$\begin{aligned} h &= \sum_{z=m}^{\infty} y^z (\sum_{i=c}^{d}\sum_{j=0}^{i} b_{i,z-i+j}(-1)^{i-j}\binom{i}{j} x_s^{d-j}x_{s+1}^j)P + \text{L.O.T.} \\ &= \sum_{z=m}^{\infty} y^z (\sum_{j=0}^{d} x_s^{d-j}x_{s+1}^j (\sum_{i=c}^{d}(-1)^{i-j}\binom{i}{j} b_{i,z-i+j}))P + \text{L.O.T.} \end{aligned}$$

The polynomial h has been written in the form $h = \sum y^z h_z$. From the properties of polynomial rings with extended coefficient fields, it follows that each $h_z \in I$. Let $r = \min\{z | \exists_i \text{ such that } b_{i,z-i} \neq 0\}$. Then, for $q = r-d, \ldots, r$,

$$h_q = \sum_{j=0}^{d} x_s^{d-j}x_{s+1}^j (\sum_{i=c}^{d}(-1)^{i-j}\binom{i}{j} b_{i,q-i+j})P + \text{L.O.T.}$$

If $j < r-q$ then $q-i+j < r-i$ and so by the definition of r

$$b_{i,q-i+j} = 0 \quad \text{for } j < r-q\ .$$

Trimming zero terms from the h_q leaves

$$h_q = \sum_{j=r-q}^{d} x_s^{d-j}x_{s+1}^j (\sum_{i=c}^{d}(-1)^{i-j}\binom{i}{j} b_{i,r-p-i+j})P + \text{L.O.T.}$$

The largest power product remaining in the expression for h_q is $x_s^{d-r+q}x_{s+1}^{r-q}P$, and it's coefficient is

$$\begin{aligned} \text{Hcoef}(h_q) &= \sum_{i=c}^{d}(-1)^{i-r+q}\binom{i}{r-q} b_{i,r-i} \\ &= \sum_{i=c}^{d}(-1)^{i-r+q}\binom{i}{r-q} \beta_i, \end{aligned}$$

where β_i denotes the value $b_{i,r-i}$. Consider the β_i to be variables. There are $d-c+1$ of these variables:

$$\beta_c, \beta_{c+1}, \ldots, \beta_d.$$

Now the $d+1$ expressions $\mathrm{Hcoef}(h_{r-d}), \ldots, \mathrm{Hcoef}(h_r)$ are linearly independent over these variables. Therefore, fixing the values of any $d-c+1$ of the $\mathrm{Hcoef}(h_k)$ has the effect of fixing the values of the β_i's. Therefore, if $d-c+1$ of these expressions evaluate to zero, then all of the β_i must be zero. This contradicts the choice of r as the least value where at least one of the β_i is non-zero. Therefore, at least $c+1$ of the $\mathrm{Hcoef}(h_k)$ are non-zero, and $\mathrm{Head}(I)$ contains at least $c+1$ different power products of the form $x_s^{d-p} x_{s+1}^p P$.
□

Corollary 5 *If $N_P(\phi_s(I'), d) = c$, then $x_s^{d-j} x_{s+1}^j P \in \mathrm{Head}(\phi_s(I'))$ if and only if $j < c$.*

Proof. By lemma (3) $N_P(I', d) = c$ and hence $N_P(I, d) = c$. If $\phi_s(I)$ contains a polynomial h whose head term is of the form $x_s^{d-j} x_{s+1}^j P$ for $j > c$, then by the previous lemma, $N_P(I, d) \geq c+1$. Thus, no such h can be in $\phi_s(I)$, so the c distinct head terms of $\phi_s(I)$ of the form $x_s^{d-j} x_{s+1}^j P$ must be exactly the ones which have $j = 0, \ldots, c-1$.
□

For monomial ideals whose coefficients lie in field K, the ideal is completely specified by the power products which they contain. In fact, the ideal can be specified by simply listing the power products of degree less than the Macaulay constant m_0 since this is certain to contain a monomial basis for the ideal.

This allows a convenient means of comparing two monomial ideals I, J of the same Hilbert function.

Define:

> $I \equiv J$ if the two ideals contain the same power products. This condition can be effectively tested by considering only those power products of degree $\leq m_0$.
>
> $I < J$ if I contains the $\underset{R}{<}$-least power product at which I and J differ. In other words, there exists an $a \in I - J$ such for all $b \in J - I$, $a \underset{R}{<} b$.

Since only monomials of degree $\leq m_0$ are under consideration, there are only a finite number of different monomial ideals for each Hilbert function.

Furthermore, this measure does not depend on the coefficient field K, and in fact allows a comparison of ideals whose coefficients lie in different fields.

Lemma 6 *Let $I \in \mathcal{A}$, and I' be the ideal generated by I in $\mathcal{A}'$. Then* $\mathrm{Head}(\phi_s(I')) \leq (\mathrm{Head}(I))$.

Proof. Assume otherwise, then there must be some $a \in \mathrm{Head}(I)$ such that

1. $a \notin \mathrm{Head}(\phi_s(I'))$, and
2. $b \in \mathrm{Head}(\phi_s(I')) - \mathrm{Head}(I) \implies a \underset{\mathrm{R}}{<} b$.

Write a as $a = x_s^{d-c}x_{s+1}^c P$ for $P \in \overline{\mathrm{PP}}_s$. If for all $j \leq c$, $x_s^{d-j}x_{s+1}^j P \in \mathrm{Head}(I)$, then $N_P(I,d) \geq c+1$, which leads to the contradiction $a \in \mathrm{Head}(\phi_s(I')$.

Otherwise, there exists a $j < c$ such that $b = x_s^{d-j}x_{s+1}^j P \notin \mathrm{Head}(I)$. Let j be the least such value. Then $N_P(I,d) \geq j+1$ and

$$\begin{aligned} b &\in \mathrm{Head}(\phi_s(I')) \\ b &\in \mathrm{Head}(\phi_s(I')) - \mathrm{Head}(I) . \end{aligned}$$

Since $b \underset{\mathrm{R}}{<} a$ this also contradicts the choice of a.
□

Lemma 7 *Let I be an ideal such that* $\mathrm{Head}(I)$ *is not Borel fixed. Then, there exists an s such that* $\mathrm{Head}(\phi_s(I')) < \mathrm{Head}(I)$.

Proof. By the previous lemma, $\mathrm{Head}(\phi_s(I')) \leq \mathrm{Head}(I)$, so it need only be shown that there exists an s such that $\mathrm{Head}(\phi_s(I'))$ and $\mathrm{Head}(I)$ are not equivalent. Since $\mathrm{Head}(I)$ is not Borel fixed, one can find an s and M ($\deg(M) < m_0$) such that $x_{s+1}M \in \mathrm{Head}(I)$, but $x_s M \notin \mathrm{Head}(I)$. Let M be written as $M = x_s^{d-c}x_{s+1}^c P$ for $P \in \overline{\mathrm{PP}}_s$. Let j be the smallest value such that $x_s^{(d+1)-j}x_{s+1}^j P \notin \mathrm{Head}(I)$. Since $x_s M \notin \mathrm{Head}(I)$, $j \leq c$. But, $N_P(I, d+1) \geq j+1$, and so $x_s^{(d+1)-j}x_{s+1}^j P \in \mathrm{Head}(\phi_s(I'))$. Therefore this power product is in one head ideal but not the other. And so, $\mathrm{Head}(I)$ and $\mathrm{Head}(\phi_s(I'))$ cannot be equivalent.
□

Lemma 8 *Let K be an infinite field. Then for any ideal I such that* Head(I) *is not Borel fixed, there exists a linear change of coordinates τ given by*

$$\tau : \begin{cases} x_{s+1} \longrightarrow kx_s + x_{s+1} & k \in K \\ x_i \longrightarrow x_i & i \neq s+1 \ . \end{cases}$$

such that Head$(\tau(I)) <$ Head(I).

Proof. By the preceding lemma there exists an s such that Head$(\phi_s(I') <$ Head(I). Let G be a reduced Gröbner basis for $\phi_s(I')$. For each $g \in G$, multiply g by an appropriate polynomial $a_g \in K[y]$ to clear the denominators so that $a_g g \in (K[y])[x_1, \ldots, x_n]$. Now the set

$$G' = \{a_g g \ : \ g \in G\}$$

is also a Gröbner basis for $\phi_s(I')$. The leading coefficients of the polynomials in G' are polynomials in y. Since K is infinite, for almost all projections $\pi_k : y \to k$ with $k \in K$, these coefficients do not become zero, and the set of polynomials

$$H = \{\pi_k(a_g g) \ : \ g \in G\}$$

will have Head$(H) =$ Head(G). Hence, Head$(pi_k(\phi_s(I'))) \supseteq$ Head$(\phi_s(I'))$, and a simple Hilbert function analysis shows that these two sets must in fact be equal. But, the composition $\pi_k \phi_s$ is simply the transformation τ, so

$$\text{Head}(\tau(I)) = \text{Head}(\pi_k(\phi_s(I')) \ < \ \text{Head}(I).$$

□

Lemma 9 *Let K be an infinite field. Then for any ideal $I \in K[x_1, \ldots, x_n]$, there exists a linear change of coordinates τ such that* Head$(\tau(I))$ *is Borel fixed.*

Proof. If Head(I) is not Borel fixed, then by the previous lemma there exists a linear change of coordinates τ_1 such that Head$(\tau_1(I)) <$ Head$)(I)$. Denote $\tau_1(I)$ by I_1. Proceeding inductively, if Head(I_m) is not Borel fixed, there exists a linear change of coordinates τ_{m+1} such that $I_{m+1} = \tau_{m+1}(I_m)$ and Head$(I_{m+1}) <$ Head(I_m). The sequence $I, I_1, I_2, \ldots$ only terminates if it reaches an ideal I_m such that Head(I_m) is Borel fixed. But, this sequence must terminate since there are only a finite number of different monomial ideals for any given Hilbert function.

□

References

[Ba 82] D. Bayer, *The Division Algorithm and the Hilbert Scheme*, **Ph.D. thesis, Harvard University 1982**.

[Bu 85] B. Buchberger, *Gröbner basis: An algorithmic method in polynomial ideal theory*, in chapter 6 of **Multidimensional Systems Theory**, (editor, N. K. Bose), D.Reidel Publishing Company, 1985, 184-229.

[Du 88] T. Dubé, *Quantitative Analysis of Problems in Computer Algebra: Gröbner Bases and the Nullstellensatz*, **Ph.D. thesis, New York University 1988**.

[Du 89] T. Dubé, *The Structure of Polynomial Ideals and Gröbner Bases* (to appear in SIAM Journal of Computing).

[Gi 84] M. Giusti, *Some Effectivity Problems in Polynomial Ideal Theory, Lecture Notes in Computer Science 174*, 1984,159-171.

[Ga 73] A. Galligo, *A Propos du Théorme de Préparation de Weierstrauss*, **Lecture Notes in Mathematics 409**, 1973, 543-579.

[MöMo 84] H.M. Möller and F. Mora, *Upper and Lower Bounds for the Degree of Groebner Bases*, **Lecture Notes in Computer Science 174**, 1984,172-183.

[ZaSa 60] O. Zariski and P. Samuel, *Commutative Algebra, Volume II*, **Springer-Verlag 1960**.

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